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Isomorphisms of the Fourier Algebras in Crossed Products

by

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Abstract.

Let $(\mathcal{A}, G, \alpha)$, $(\mathcal{B}, H, \beta)$ be W*-systems, $F_\alpha(G;\mathcal{A}_*)$ and $F_\beta(H;\mathcal{B}_*)$, their Fourier algebras defined in [2]. The main result is that $F_\alpha(G;\mathcal{A}_*)$ and $F_\beta(H;\mathcal{B}_*)$ are isometrically isomorphic as Banach algebras if and only if either G and H are topologically isomorphic (denoted by I) as groups and $\mathcal{A}$ and $\mathcal{B}$ are isomorphic (denoted by θ) such that $\beta_{I(g)}\circ\theta = \theta\circ\alpha_g$ for all $g \in G$, or G and H are topologically anti-isomorphic and $\mathcal{A}$ and $\mathcal{B}$ are anti-isomorphic such that $\beta_{I(g^{-1})}\circ\theta = \theta\circ\alpha_g$ for all $g \in G$.

1. Introduction.

For locally compact abelian groups G, H, Pontryagin's duality theorem mentions that $L^1(G)$ and $L^1(H)$ are isometrically isomorphic if and only if G and H are topologically isomorphic as groups. T. Kawada [4] and J.G. Wendel [11] proved the above statement for arbitary locally compact groups.

G is a locally compact abelian group, then $L^1(G)$ is isometrically isomorphic to Fourier algebra A(G) in [7]. Therefore A(G) and A(H) are isometrically isomorphic as Banach algebras if and only if G and H are topologically isomorphic as abelian groups.

P. Eymard [1], on the other hand, defined the Fourier algebra A(G) of a locally compact group G and showed that A(G) is isometrically isomorphic to the predual $m(G)_*$ of the von Neumann algebra m(G) generated by the left regular representation of G.

So that, M.E.Walter [10] showed that A(G) and A(H) are isometrically isomorphic as Banach algebras if and only if G and H are topologically isomorphic as groups for arbitary locally compact groups.

Recently for W*-system $(\mathfrak{A}, G, \alpha)$, the Fourier space $F_\alpha(G;\mathfrak{A}_*) \subset C_0(G:\mathfrak{A}_*)$ was defined in [8] H. Takai such that $F_\alpha(G;\mathfrak{A}_*)$ is isometrically isomorphic to the predual of the crossed product $G \otimes_\alpha \mathfrak{A}$ as Banach spaces.

M. Fugita [2] quite recently defined the Banach algebra structure in Fourier space $F_\alpha(G;\mathfrak{A}_*)$ and all characters $\widehat{F_\alpha(G;\mathfrak{A}_*)}$ is topologically isomorphic to G as groups and defined and investigated the support of the operators in $G \otimes_\alpha \mathfrak{A}$.

In this paper we generalize a Walter's result for W*-system $(\mathfrak{A}, G, \alpha)$ and show that the Banach algebra structure in $F_\alpha(G;\mathfrak{A}_*)$ is essential in a sence.

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2. Notations and Preliminaries.

Let $\mathcal{O}\!\mathcal{L}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and G be a locally compact group. The triple $(\mathcal{O}\!\mathcal{L}, G, \alpha)$ is said a W^*-system if the mapping α of G into the group $\mathrm{Aut}(\mathcal{O}\!\mathcal{L})$ of all automorphisms of $\mathcal{O}\!\mathcal{L}$ is a homomorphism and the function $g \mapsto \omega\circ\alpha_g(x)$ is continuous on G for all $x \in \mathcal{O}\!\mathcal{L}$ and $\omega \in \mathcal{O}\!\mathcal{L}_*$ ($\mathcal{O}\!\mathcal{L}_*$ is the predual of $\mathcal{O}\!\mathcal{L}$).

$G \otimes_\alpha \mathcal{O}\!\mathcal{L}$ is the von Neumann algebra generated by the family of the operators $\{\pi_\alpha(x), \lambda_G(g) \ ; \ x \in \mathcal{O}\!\mathcal{L}, \ g \in G\}$;

$$(\pi_\alpha(x)\xi)(h) = \alpha_h^{-1}(x)\xi(h)$$

$$(\lambda_G(g)\xi)(h) = \xi(g^{-1}h)$$

for $\xi \in L^2(G;\mathcal{H})$.

Each element ω of the predual $(G \otimes_\alpha \mathcal{O}\!\mathcal{L})_*$ of $G \otimes_\alpha \mathcal{O}\!\mathcal{L}$ may be regarded as an element u_ω of $C^b(G, \mathcal{O}\!\mathcal{L}_*)$;

$$u_\omega[g](x) = \langle \pi_\alpha(x)\lambda(g), \omega \rangle$$

for all $x \in \mathcal{O}\!\mathcal{L}$, $g \in G$ where $C^b(G;\mathcal{O}\!\mathcal{L}_*)$ is the space of all bounded continuous functions. And the new norm $\|\ \ \|$ on $F_\alpha(G;\mathcal{O}\!\mathcal{L}_*)$ is defined ;

$$\| u_\omega \| = \| \omega \|$$

such that $\| u \|_\infty \leq \| u \|$ for all $u \in F_\alpha(G;\mathcal{O}\!\mathcal{L}_*)$ where $F_\alpha(G;\mathcal{O}\!\mathcal{L}_*) = \{u_\omega \ ; \ \omega \in (G \otimes_\alpha \mathcal{O}\!\mathcal{L})_*\} \subset C^b(G;\mathcal{O}\!\mathcal{L}_*)$.

We define the product on $F_\alpha(G;\mathcal{O}\!\mathcal{L}_*)$ by ;

$$(u * v)[g](x) = u(g)(x)v(g)(1)$$

for all $u, v \in F_\alpha(G;\mathcal{O}\!\mathcal{L}_*)$, $x \in \mathcal{O}\!\mathcal{L}$, $g \in G$. Then $F_\alpha(G;\mathcal{O}\!\mathcal{L}_*)$ became a Banach algebra ([2] Theorem 3.5). So that we can define the product with an operator T in $G \otimes_\alpha \mathcal{A}$ and an element u in $F_\alpha(G;\mathcal{A}_*)$;

$$< uT, v > = < T, v * u >$$

$$< Tu, v > = < T, u * v >$$

for all $v \in F_\alpha(G;\mathcal{A}_*)$ ((3.7), (3.9) in [2]).

Let T be an operator in $G \otimes_\alpha \mathcal{A}$. Then the support supp(T) of T is the set of all $g \in G$ satisfying the condition that $\lambda_G(g)$ belongs to the σ-weak closure of $TF_\alpha(G;\mathcal{A}_*)$ [See [2] Proposition 4.1].

Theorem. Let $(\mathcal{A}, G, \alpha)$, $(\mathcal{B}, H, \beta)$ be W^*-systems and $F_\alpha(G;\mathcal{A}_*)$, $F_\beta(H;\mathcal{B}_*)$ their Fourier algebras. Let ϕ be an isometric isomorphism of $F_\alpha(G;\mathcal{A}_*)$ onto $F_\beta(H;\mathcal{B}_*)$ as Banach algebras.

Then we get five elements (k, p, q, I, θ) with the following properties;

(1) k is an element of G such that $\lambda_G(k) = {}^t\phi(\lambda_H(e))$, where ${}^t\phi$ is the transposed map of ϕ, e is the identity of H,

(2) I is either an isomorphism or anti-isomorphism of H onto G as locally compact groups,

(3) p(resp. q) is a projection of $\mathcal{Z}_{\mathcal{A}} \cap \mathcal{A}^G$ (resp. $\mathcal{Z}_{\mathcal{B}} \cap \mathcal{B}^H$),

(4) θ is a isometric linear map of $\mathcal{B}$ onto $\mathcal{A}$ such that,

θ is an isomorphism of $\mathcal{B}_q$ onto $\mathcal{A}_p$,

θ is an anti-isomophism of $\mathcal{B}_{1-q}$ onto $\mathcal{A}_{1-p}$,

(5) $\phi(u)[h](y) = ({}_ku)[I(h)](\theta(y)p) + ({}_ku)[I(h)](\alpha_{I(h)}(\theta(y))(1-p))$

for all $y \in \mathcal{B}$, $h \in H$ and $u \in F_\alpha(G;\mathcal{A}_*)$, where $({}_ku)[g](y) = u[kg](\alpha_k(y))$,

(6) $\theta[\beta_k(y)] = [\alpha_{I(h)} \circ \theta(y)]p + [\alpha^{-1}_{I(h)} \circ \theta(y)](1-p)$

for all $y \in \mathcal{B}$, $h \in H$.

Corollary. Let $(\mathfrak{A}, G, \alpha)$, $(\mathfrak{B}, H, \beta)$ be W^*-systems, the two actions α and β are ergodic on their centers (ie. $\mathfrak{Z}_{\mathfrak{A}} \cap \mathfrak{A}^G = \mathfrak{Z}_{\mathfrak{B}} \cap \mathfrak{B}^H = \mathbb{C}$).

The following statements are equivalent ;

(i) $F_\alpha(G;\mathfrak{A}_*) \cong F_\beta(H;\mathfrak{B}_*)$ in the sense of Banach algebras,

(ii) there exist either an isomorphism I of H onto G, an isomorphism θ of $\mathfrak{B}$ onto $\mathfrak{A}$ such that $\theta \circ \beta_h = \alpha_{I(h)} \circ \theta$ for all $h \in H$, or an anti-isomorphism I of H onto G, an anti-isomorphism θ of $\mathfrak{B}$ onto $\mathfrak{A}$ such that $\theta \circ \beta_h = \alpha_{I(h^{-1})} \circ \theta$ for all $h \in H$.

[The proof of Theorem]. The transposed map ${}^t\phi$ of ϕ is an isometric linear map of $H \otimes_\beta \mathfrak{B}$ onto $G \otimes_\alpha \mathfrak{A}$. Using [3] Theorem 7, 10, we get ;

$${}^t\phi = {}^t\phi(\lambda_H(e))(\gamma_I + \gamma_A)$$

where γ_I is an isomorphism of $(H \otimes_\beta \mathfrak{B})_{z'}$ onto $(G \otimes_\alpha \mathfrak{A})_z$, γ_A is an anti-isomorphism of $(H \otimes_\beta \mathfrak{B})_{(1-z')}$ onto $(G \otimes_\alpha \mathfrak{A})_{(1-z)}$, z(resp. z') is a central projection of $G \otimes_\alpha \mathfrak{A}$ (resp. $H \otimes_\beta \mathfrak{B}$).

For all $u, v \in F_\alpha(G;\mathfrak{A}_*)$, $h \in H$, we obtain ;

$$\begin{aligned} \langle {}^t\phi(\lambda_H(h)), u * v \rangle &= \langle \lambda_H(h), \phi(u * v) \rangle \\ &= \langle \lambda_H(h), \phi(u) * \phi(v) \rangle \\ &= \langle \lambda_H(h) \otimes \lambda_H(h), \phi(u) \otimes \phi(v) \rangle \\ &= \langle {}^t\phi(\lambda_H(h)), u \rangle \langle {}^t\phi(\lambda_H(h)), v \rangle . \end{aligned}$$

Therefore ${}^t\phi(\lambda_H(h))$ is a character of $F_\alpha(G;\mathfrak{A}_*)$ for all $h \in H$, which implies ${}^t\phi(\lambda_H(H)) = \lambda_G(G)$ since the character space $F_\alpha(G;\ _*)$ is isomorphic to G ([2] theorem 3.14).

We denote ${}^t\phi(\lambda_H(e))$ by $\lambda_G(k)$.

By the quite same arguement in [10] Theorem 2 we get that

$$\gamma \equiv {}^t\phi(\lambda_H(e))^{-1}\, {}^t\phi = \gamma_I + \gamma_A$$

is C^*-isomorphism in Kadison's sense [3] and $\gamma(\lambda_H(h_1)\lambda_H(h_2))$ is either $\gamma(\lambda_H(h_1))\ \gamma(\lambda_H(h_2))$ or $\gamma(\lambda_H(h_2)\ \gamma(\lambda_H(h_1))$, moreover we put $\gamma(\lambda_H(h)) = \lambda_G(I(h))$, so that I is either an isomorphism or an anti-isomorphism of H onto G as locally compact groups.

The transposed map ψ of γ is also an isometric isomorphism of $F_\alpha(G;\mathfrak{A}_*)$ onto $F_\beta(H;\mathfrak{B}_*)$. Then we get ;

$$\begin{aligned}
< \gamma(\pi_\beta(y)),\ u * v > &= < \pi_\beta(y),\ \psi(u * v) > \\
&= < \pi_\beta(y),\ \psi(u) * \psi(v) > \\
&= < \pi_\beta(y) \otimes 1,\ \psi(u) \otimes \psi(v) > \\
&= < \gamma(\pi_\beta(y)) \otimes 1,\ u \otimes v >
\end{aligned}$$

for all $y \in \mathfrak{B}$, $u, v \in F_\alpha(G;\mathfrak{A}_*)$.

By [5] proposition 2.3, we obtain $\gamma(\pi_\beta(y))$ is an element of $\pi_\alpha(\mathfrak{A})$, so that we can define a isometric surjective linear map θ of $\mathfrak{B}$ onto $\mathfrak{A}$ by $\theta = \pi_\alpha^{-1} \circ \gamma \circ \pi_\beta$.

Since γ is a Jordon isomorphism,

$$\gamma(T)\gamma(z') + \gamma(z')\gamma(T) = \gamma([T, z']) = 2\gamma(T\ z')$$

for all $T \in H \otimes_\beta \mathfrak{B}$, therefore we get $\gamma(T\ z') = \gamma(T)z$.

Hence $$\gamma(\pi_\beta(x\ y))z = \gamma(\pi_\beta(x))\ \gamma(\pi_\beta(y))z$$

for all $x, y \in \mathfrak{B}$.

Since z is a central projection of $G \otimes_\alpha \mathfrak{A}$, z is also an projection of $\pi_\alpha(\mathfrak{A})'$, so that $\gamma(\pi_\beta(x\,y))p = \gamma(\pi_\beta(x))\gamma(\pi_\beta(y))p$ for all $x,y \in \mathfrak{B}$ where p is the central support of z in the von Neumann algebra $\pi_\alpha(\mathfrak{A})'$.

We denote by q the central support of z' in the von Neumann algebra $\pi_\beta(\mathfrak{B})'$, then $\gamma(q)z = \gamma(q\,z') = \gamma(z') = z$, implies that $\gamma(q)p = p$, similarly we also obtain $\gamma^{-1}(p)q = q$ so that $\gamma(q) = \gamma(\gamma^{-1}(p)q) = \gamma(\gamma^{-1}(p))\ \gamma(q)p = p\ \gamma(q)p = p$.

Hence θ is an isomorphism of $\mathfrak{B}_q$ onto $\mathfrak{A}_p$, moreover by the quite same arguement, θ is an anti-isomorphism of $\mathfrak{B}_{1-q}$ onto $\mathfrak{A}_{1-p}$.

Since $\pi_\alpha(\mathfrak{A})' = \lambda_G(g)\ \pi_\alpha(\mathfrak{A})'\ \lambda_G(g)^*$, $\lambda(g)z\lambda(g)^* = z$ for all $g \in G$, we can prove easily that p is a G-invariant projection of $\mathfrak{A}$, similarly q is a H-invariant projection of $\mathfrak{B}$.

Now we have already proved (1) $\sim$ (4) and the statements (5) and (6) still remain.

For all $y \in \mathfrak{B}$, $h \in H$, we get

$$\begin{aligned}\{\pi_\alpha \circ \theta(\beta_h(y))\}z &= \gamma(\lambda_H(h)\pi_\beta(y)\lambda_H(h)^* z') \\ &= \lambda_G(I(h))\pi_\alpha\theta(y)\lambda_G(I(h)^{-1})z \\ &= \pi_\alpha\ \alpha_{I(h)}\ \theta(y)z.\end{aligned}$$

Hence we get $\theta \circ \beta_h = \alpha_{I(h)} \circ \theta$ on $\mathfrak{B}_q$, and similarly

$\theta \circ \beta_h = \alpha_{I(h)^{-1}} \circ \theta$ on $\mathfrak{B}_{1-q}$ for all $h \in H$.

Therefore we get ;

$$\theta \circ \beta_h(y) = \alpha_{I(h)} \circ \theta(y)p + \alpha_{I(h)^{-1}} \circ \theta(y)(1-p)$$

for all $y \in \mathfrak{B}$ and $h \in H$.

To prove the statement (5), we shall show first ;

$$\operatorname{supp} \gamma(\pi_\beta(y)\lambda_H(h)) = \{I(h)\} .$$

For $u \in F_\alpha(G;\mathfrak{A}_*)$, since $(\gamma(\pi_\beta(y)\ \lambda_H(h)))u = \gamma(\pi_\beta(y)\ \lambda_H(h)\psi(u))$ and ψ is surjective, we get ;

$$\overline{[\gamma(\pi_\beta(y)\ \lambda_H(h))\ F_\alpha(G;\mathfrak{A}_*)]}^{\sigma-w}$$

$$= \gamma\overline{[\pi_\beta(y)\ \lambda_H(h)\ F_\beta(H;\mathcal{B}_*)]}^{\sigma-w} ,$$

therefore $\overline{[\pi_\beta(y)\ \lambda_H(h)\ F_\beta(H;\mathcal{B}_*)]}^{\sigma-w} \cap \lambda_H(H) = \mathbb{C}\ \lambda_H(h)$ because of $\operatorname{supp} \pi_\beta(y)\lambda_H(h) = \{h\}$, so that we obtain ;

$$\overline{[\gamma(\pi_\beta(y)\ \lambda_H(h))F_\alpha(G;\mathfrak{A}_*)]}^{\sigma-w} \cap \lambda_G(G) = \mathbb{C}\ \lambda_G(I(h)) ,$$

that is $\operatorname{supp} \gamma(\pi_\beta(y)\ \lambda_H(h)) = \{I(h)\}$.

By [2] Theorem 4.4 or [6] Proposition 6.1, there exists an element x of $\mathfrak{A}$ such that $\gamma(\pi_\beta(y)\ \lambda_H(h)) = \pi_\alpha(x)\ \lambda_G(I(h))$.

On the other hand,
$$\begin{aligned}\pi_\alpha(x)\ \lambda_G(I(h))z &= \gamma(\pi_\beta(y)\ \lambda_H(h))z \\ &= \gamma(\pi_\beta(y))\ \gamma(\lambda_H(h))z \\ &= \pi_\alpha(\theta(y))\ \lambda_G(I(h))z\end{aligned}$$

therefore, we get $x\,p = \theta(y)p$, similarly we obtain $x(1-p) = \alpha_{I(h)} \circ \theta(y)(1-p)$, hence $x = \theta(y)p + \alpha_{I(h)} \circ \theta(y)(1-p)$,

$$\gamma(\pi_\beta(y)\ \lambda_H(h)) = \pi_\alpha(\theta(y)p)\ \lambda_G(I(h)) + \pi_\alpha(\alpha_{I(h)} \circ \theta(y)(1-p))\lambda_G(I(h)) .$$

By the definition of Fourier algebras, the above equation and $\phi(u) = \psi({}_k u)$ for all $u \in F_\alpha(G;\mathfrak{A}_*)$, we can get the statement (5) easily.

[Proof of Cor.] Suppose ϕ is an isometric isomophism of $F_\alpha(G;\mathfrak{A}_*)$ onto $F_\beta(H;\mathfrak{B}_*)$ and we use the same notations in the proof of the Theorem. The projection p in the Theorem must be zero or 1 by the conditions in the corollary, therefore θ must be either an isomorphism or an anti-isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$.

When G is a locally compact abelian group, (which implies that H is also a locally compact abelian group), I can be regarded as both an isomorphism and an anti-isomorphism as we like, therefore the Theorem says that θ is either an isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ such that I is an isomorphism of h onto G and $\alpha_{I(h)}\circ\theta = \theta\circ\beta_h$ for all $h \in H$, or an anti-isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ such that I is anti-isomorphic and $\alpha_{I(h)}^{-1}\circ\theta = \theta\circ\beta_h$ for all $h \in H$. Hence we may assume that G is non-abelian. When I is an anti-isomorphism of H onto G, the projection(1-z) appearing in the proof of the Theorem must be non-zero. For, if the projection z is an identity operator in $G\otimes_\alpha\mathfrak{A}$ then γ is an isomorphism of $H\otimes_\beta\mathfrak{B}$ onto $G\otimes_\alpha\mathfrak{A}$, so that the arguement in the construction of the anti-isomorphism I tell us that I is isomorphic [See [10] Theorem 2]. Then I is both anti-isomorphic and isomorphic, which implies that G is an abelian group, which is a contradition. Then we have gotten the projection (1-z) is non-zero. Instead of considering the central support p of z in the proof of the Theorem, we may take the central support of (1-z) in the von Neumann algebra $\pi_\alpha(\mathfrak{A})'$, hence θ must be anti-isomorphic such that $\alpha_{I(h^{-1})}\circ\theta = \theta\circ\beta_h$ for all $h \in H$. If I is an isomorphism of H onto G, we similarly get the conclusion that θ is isomorphic such that $\alpha_{I(h)}\circ\theta = \theta\circ\beta_h$ for all $h \in H$.

Conversely, suppose I is an isomorphism of H onto G such that $\theta\circ\beta_h = \alpha_{I(h)}\circ\beta_h$ for all $h \in H$. [9] proposition 3.4 says that there exists an isomorphism γ of $H \otimes_\beta \mathcal{B}$ onto $G \otimes_\alpha \mathfrak{A}$ such that $\gamma(\pi_\beta(y)) = \pi_\alpha(\theta(y))$ for all $y \in \mathcal{B}$, $\gamma(\lambda_H(h)) = \lambda_G(I(h))$ for all $h \in H$.

Then the transposed map ϕ of γ is an isometric isomorphism of $F_\alpha(G;\mathfrak{A}_*)$ onto $F_\beta(H;\mathcal{B}_*)$.

Suppose I is an anti-isomorphism of H onto G such that $\theta \circ \beta_h = \alpha_{I(h^{-1})} \circ \theta$ for all $h \in H$. By considering the opposite von Neumann algebra $\mathfrak{A}^0$ of $\mathfrak{A}$, the isomorphism J of H onto G by $J(h) = I(h^{-1})$ for all $h \in H$, similarly above, there exists an isomorphism γ of $H \otimes_\beta \mathcal{B}$ onto $G \otimes_\alpha \mathfrak{A}^0$ such that $\gamma(\pi_\beta(y)) = \pi_\alpha(\theta(y))$ for all $y \in \mathcal{B}$, $\gamma(\lambda_H(h)) = \lambda_G(J\, h)$ for all $h \in H$. On the other hand $G \otimes_\alpha \mathfrak{A}^0$ is isomorphic to $G \otimes_\alpha \mathfrak{A}$ as Banach spaces, therefore there exists an isometric linear map γ of $H \otimes_\beta \mathcal{B}$ onto $G \otimes_\alpha \mathfrak{A}$ with the above properties. Then it is quite clear that the transposed map ϕ of γ is an isometric isomorphism of $F\ (G;\mathfrak{A}_*)$ onto $F_\beta(H;\mathcal{B}_*)$.

Remark 1. This theorem is a kind of the generalization of [10] Theorem 2 .

Remark 2. Let $(\mathfrak{A}, G, \alpha)$, $(\mathfrak{A}, G, \beta)$ be W^*-systems. Then the algebraic tensor product $A(G) \odot \mathfrak{A}_*$ with the Fourier algebra $A(G)$ of G and the predual $\mathfrak{A}_*$ of $\mathfrak{A}$ is naturally imbeded in both the Fourier algebras $F_\alpha(G;\mathfrak{A}_*)$ $F_\beta(G;\mathfrak{A}_*)$, moreover $A(G) \odot \mathfrak{A}_*$ is dense in these, therefore if the identity map i of $A(G) \odot \mathfrak{A}_* \subset F_\alpha(G;\mathfrak{A}_*)$ onto $A(G) \odot \mathfrak{A}_* \subset F_\beta(G;\mathfrak{A}_*)$ can be extended isometrically from $F_\alpha(G;\mathfrak{A}_*)$ onto $F_\beta(G;\mathfrak{A}_*)$, the two actions α, β are quite same in a sence. The algebric structure of $F_\alpha(G;\mathfrak{A}_*)$ determines the group structure of G and the norm in $F_\alpha(G;\mathfrak{A}_*) \subset C_0(G;\mathfrak{A}_*)$ which is quite different from the sup-norm in $C^b(G;\mathfrak{A}_*)$ determines the action α of G.

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